Relativity writeups

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1 Introduction

This is a summary of all the work our group has done through the last several months in our independent study on Special and General Relativity.

2 Center of Mass and Lab Frame

2.1 Boosting from Lab Frame to CoM

Suppose we have a system whose four-momentum is $\begin{bmatrix} E_{lab}/c \\ \vec{P} \end{bmatrix}$ and we would like to boost this to a frame where there is no spatial component $\begin{bmatrix} E_{CoM}/c \\ 0 \end{bmatrix}$. As the magnitude of the four-momentum is invariant under a Lorentz boost,

$$-(E_{lab}/c)^2 + \vec{P} \cdot \vec{P} = -(E_{CoM}/c)^2 \tag{1}$$

$$\implies E_{CoM}^2 = E_{lab}^2 - c^2 \vec{P} \cdot \vec{P} \tag{2}$$

2.2 Why scattering between two moving particles is more energy efficient?

Suppose, in lab frame, particle A has energy $E_{ALF} = \gamma_A M_A c^2$. If one wants to increase its energy until the total CoM energy has increased by a factor of λ for particle production through scattering with another particle B, let's find the new lab frame energy, E_N. First off, we can see that for all particles in this process:

$$\gamma_i = \frac{E_i}{M_i c^2} \implies v_i^2 = c^2 - \frac{c^2}{\gamma_i^2} = c^2 - \frac{M_i c^4}{E_i} \tag{3}$$

$$\implies \vec{P}_i \cdot \vec{P}_i = \frac{E_{ALF}^2}{c^4} \vec{v}_i \cdot \vec{v}_i \tag{4}$$

We can calculate the system CoM energy from its lab frame energy via:

$$E_{CoM}^2 = (E_{ALF} + M_B c^2)^2 - c^2 \vec{P}_{ALF} \cdot \vec{P}_{ALF} \tag{5}$$

$$= (E_{ALF} + M_B c^2)^2 - \frac{E_{ALF}^2}{c^2} v_A^2 \tag{6}$$

We want a new CoM energy, whose lab frame energy E_N can be calculated similarly from above, to be the same as λ times the old lab frame energy,

$$\lambda^2 \left((E_{ALF} + M_B c^2)^2 - \frac{E_{ALF}^2}{c^2} v_A^2 \right) = (E_N + M_B c^2)^2 - \frac{E_N^2}{c^2} v_N^2 \tag{7}$$

We can rewrite the velocities in terms of their corresponding energies using the relation above to get an equation with just E_{AFL}, E_N, the masses, and c. We can then solve for E_N in terms of everything else (note that after doing this the square energies from the parenthesis cancel out with that from the four-momentum square so we get a nice linear equation) :

$$E_N = \frac{-c^2 M_B^2 + E_{ALF} M_A \lambda^2 + 2E_{ALF} M_B \lambda^2 + c^2 M_B \lambda^2}{M_A + 2M_B} \tag{8}$$

Suppose we want to double the energy during a scattering process with two identical particles, this simplifies to:

$$E_N = \frac{12E_{ALF} M_A + 3c^2 M_A^2}{3M_A^2} = 4E_{ALF} + c^2 M_A^2 \approx 4E_{ALF} \tag{9}$$

3 Compton Scattering

Compton scattering is a phenomenon in which a photon collides with a stationary electron and transfers some of its energy to the electron. To derive the formula for the Compton shift, we use the principles of special relativity. In relativity, momentum is described by four-vectors, which include both the spatial components of momentum and the temporal component of energy.

Four-momentum vector of the electron before collision:

$$P_e^\mu = \begin{pmatrix} Mc \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Four-momentum vector of the proton before collision:

$$P_p^\mu = \begin{pmatrix} \mathrm{E}/c \\ \mathrm{E}/c \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

Four-momentum vector of the electron after collision:

$$P_e^{\mu\prime} = \begin{pmatrix} \gamma mc \\ \gamma Mv\cos\theta_e \\ \gamma Mv\sin\theta_e \\ 0 \end{pmatrix} \tag{12}$$

Four-momentum vector of the proton after collision:

$$P_p^{\mu\prime} = \begin{pmatrix} E'/c \\ (E'/c)\cos\theta_p \\ (-E'/c)\sin\theta_p \\ 0 \end{pmatrix} \tag{13}$$

Then using the conservation of momentum in relativity, we can relate the four-momentum vectors of the particles before and after the collision. In particular, the sum of the four-momentum vectors of the particles must be conserved in any inertial frame of reference.

$$P_e^\mu + P_p^\mu = P_e^{\mu'} + P_p^{\mu'} \tag{14}$$

Substituting the four-momentum vectors into the conservation equation, we get:

$$\begin{pmatrix} Mc \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E/c \\ E/c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma mc \\ \gamma Mv\cos\theta_e \\ \gamma Mv\sin\theta_e \\ 0 \end{pmatrix} + \begin{pmatrix} E'/c \\ E'/c\cos\theta_p \\ -E'/c\sin\theta_p \\ 0 \end{pmatrix} \tag{15}$$

Equating the components of the four-momentum vectors, we obtain:

$$E/c + Mc = \gamma mc + E'/c \tag{16}$$

$$E/c = \gamma Mv\cos\theta_e + (E'/c)\cos\theta_p \tag{17}$$

$$0 = \gamma Mv\sin\theta_e - (E'/c)\sin\theta_p \tag{18}$$

From the first equation, we can get the expression for γ:

$$\gamma = \frac{E' + Mc^2 - E}{mc} \tag{19}$$

From the third equation, we can get an expression for E':

$$E' = \gamma Mv\sin\theta_e \frac{c}{\sin\theta_p} \tag{20}$$

Now, substitute the expression for γ from the first equation into the expression for E':

$$E' = \left(\frac{E' + Mc^2 - E}{mc}\right) Mv\sin\theta_e \frac{c}{\sin\theta_p} \tag{21}$$

Now, let's use the second equation to eliminate γ Mv:

$$\gamma Mv\cos\theta_e = \frac{E}{c} - \frac{E'}{c}\cos\theta_p \tag{22}$$

Divide both sides by $\cos\theta_e$:

$$\gamma Mv = \frac{E}{c\cos\theta_e} - \frac{E'}{c}\frac{\cos\theta_p}{\cos\theta_e} \tag{23}$$

Now, substitute this expression for γ Mv into the expression for E':

$$E' = \left(\frac{E' + Mc^2 - E}{mc}\right)\left(\frac{E}{c\cos\theta_e} - \frac{E'}{c}\frac{\cos\theta_p}{\cos\theta_e}\right)\sin\theta_e \frac{c}{\sin\theta_p} \tag{24}$$

Simplify the equation:

$$E' = \frac{E' + Mc^2 - E}{m}\left(\frac{E}{\cos\theta_e} - \frac{E'}{\cos\theta_e}\cos\theta_p\right)\sin\theta_e \frac{1}{\sin\theta_p} \tag{25}$$

Multiply both sides by $\sin\theta_p$:

$$E' \sin\theta_p = \frac{E' + Mc^2 - E}{m} \left(\frac{E}{\cos\theta_e} - \frac{E'}{\cos\theta_e} \cos\theta_p \right) \sin\theta_e \tag{26}$$

Now, expand the equation and rearrange the terms to isolate E':

$$E' \sin\theta_p = \frac{E(E' + Mc^2 - E)\sin\theta_e}{m\cos\theta_e} - \frac{E'^2 \sin\theta_e \cos\theta_p}{m\cos\theta_e} \tag{27}$$

Now, factor out E' from the right side:

$$E' \sin\theta_p = \frac{E' \sin\theta_e}{m\cos\theta_e}(E - E'\cos\theta_p + Mc^2) \tag{28}$$

Now, divide both sides by $\sin\theta_e/(\text{msin}\cos_e)$:

$$E' = \frac{EMc^2}{E(1-\cos\theta_p) + Mc^2} \tag{29}$$

We have derived the expression for E' in terms of E, M, c, and $\cos\theta_p$.To find the change in wavelength, $\lambda' - \lambda$, we will use the given expression:

$$\lambda' - \lambda = \frac{hc}{E'} - \frac{hc}{E} \tag{30}$$

We already derived the expression for E':

$$E' = \frac{EMc^2}{E(1-\cos\theta_p) + Mc^2} \tag{31}$$

Now, let's plug this expression for E' into the change in wavelength equation:

$$\lambda' - \lambda = \frac{hc}{\frac{EMc^2}{E(1-\cos\theta_p)+Mc^2}} - \frac{hc}{E} \tag{32}$$

To simplify, we can factor out hc:

$$\lambda' - \lambda = hc\left(\frac{1}{E'} - \frac{1}{E}\right) \tag{33}$$

Now, plug in the expression for E' again:

$$\lambda' - \lambda = hc\left(\frac{1}{\frac{EMc^2}{E(1-\cos\theta_p)+Mc^2}} - \frac{1}{E}\right) \tag{34}$$

To simplify, let's invert the fractions:

$$\lambda' - \lambda = hc\left(\frac{E(1-\cos\theta_p) + Mc^2}{EMc^2} - \frac{E}{E}\right) \tag{35}$$

Now, let's find a common denominator (EMc^2):

$$\lambda' - \lambda = hc\left(\frac{E^2(1-\cos\theta_p) + EMc^2}{E^2Mc^2} - \frac{EMc^2}{E^2Mc^2}\right) \tag{36}$$

Now, subtract the fractions:

$$\lambda' - \lambda = hc\left(\frac{E^2(1-\cos\theta_p) + EMc^2 - EMc^2}{E^2Mc^2}\right) \tag{37}$$

Simplify the numerator:

$$\lambda' - \lambda = hc\left(\frac{E^2(1-\cos\theta_p)}{E^2Mc^2}\right) \tag{38}$$

Now, cancel the E^2 terms:

$$\lambda' - \lambda = \frac{hc(1-\cos\theta_p)}{Mc^2} \tag{39}$$

This is the expression for the change in wavelength, $\lambda' - \lambda$, in terms of h, c, M, and $\cos(\theta_p)$.

This formula shows that the change in wavelength is proportional to the cosine of the electron scattering angle and inversely proportional to the mass of the electron, and also depends on the cosine of the proton scattering angle and inversely proportional to the energy of the incident photon. This formula agrees well with experimental data and provides a more complete understanding of the scattering process.

4 Calculus of Variation

4.1 Newtonian potential

By the argument for the equivalence principle given in chapter 6 of Hartle, signals can have their rate affected by a difference in potential according to:

$$\text{rate at B} = \left(1 + \frac{\Phi_A - \Phi_B}{c^2}\right) \times \text{rate at A} \tag{40}$$

To see this in application, let's look at the geometry described by a small potential dependent in our line element $\Phi(x^i)$ for $i = x, y, z$

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)(cdt)^2 + \left(1 - \frac{2\Phi}{c^2}\right)(d\vec{x}\cdot d\vec{x}) \tag{41}$$

$$d\tau^2 = -\left(1 + \frac{2\Phi}{c^2}\right)dt^2 + \frac{1}{c^2}\left(1 - \frac{2\Phi}{c^2}\right)(d\vec{x}\cdot d\vec{x}) \tag{42}$$

for $d\vec{x} = \frac{d\vec{x}}{dt}dt = \vec{v}dt$ let's compute the total proper time:

$$\tau_{AB} = \int_A^B d\tau \tag{43}$$

$$= \int_A^B dt\sqrt{\left(1 + \frac{2\Phi}{c^2}\right) - \left(1 - \frac{2\Phi}{c^2}\right)\frac{\vec{v}\cdot\vec{v}}{c^2}} \tag{44}$$

Taylor expands this around the small-expression after the 1 + and drop all the second order terms and higher power of c gives:

$$= \int_A^B dt\left(1 + \frac{\Phi}{c^2} - \frac{1}{2}\frac{\vec{v}\cdot\vec{v}}{c^2}\right) \tag{45}$$

extremize the proper time using Euler-Lagrange equation gives:

$$\frac{\partial L}{\partial \vec{x}} = \frac{\partial}{\partial t}\frac{\partial L}{\partial \vec{v}} \tag{46}$$

$$\implies \frac{1}{c^2}\nabla\Phi - \frac{\partial}{\partial t}\left(\frac{-\vec{v}}{c^2}\right) = 0 \tag{47}$$

$$\nabla\Phi + \frac{\partial^2 \vec{x}}{\partial t^2} = 0 \tag{48}$$

which looks a lot like Newton's second law, so we do recover classical mechanics for a small potential that only depends on the position.

4.2 Principle of least action

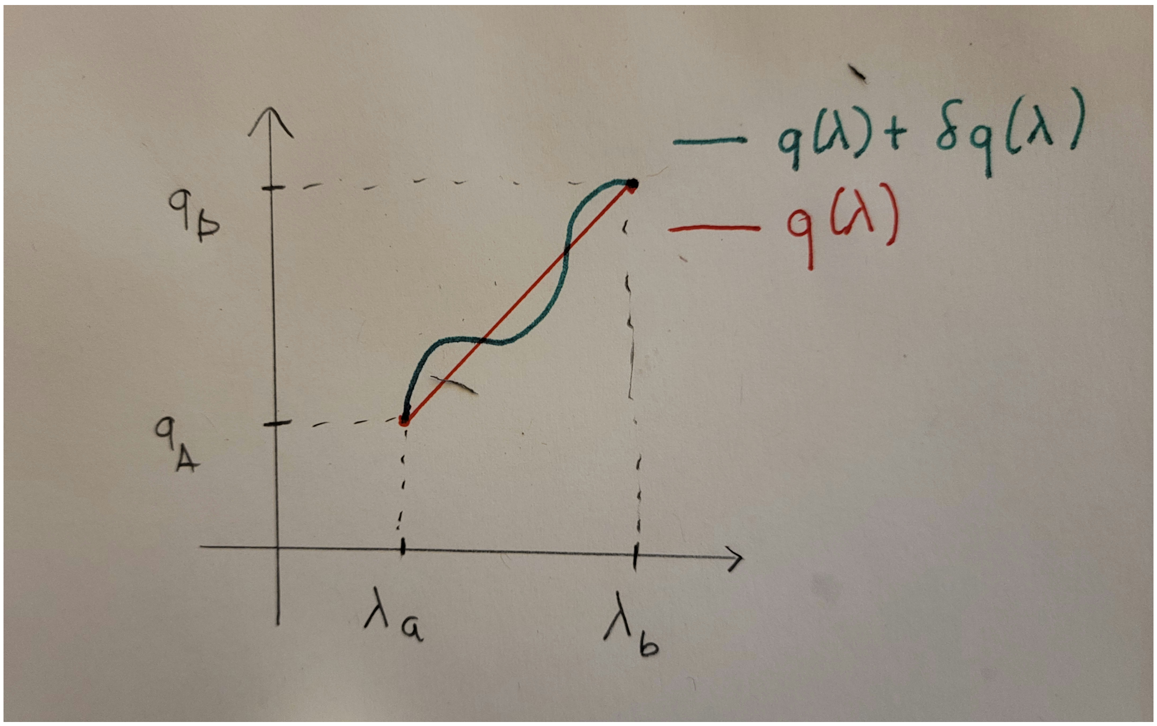


Figure 1: $q(\lambda)$ is the path that minimizes our action, the other path has a small deviation $\delta q(\lambda)$

Let's look at the action functional $I[q]$ that takes any parametrization of any path $f(x) \mapsto \mathbb{C}$ through the following map:

$$I[q] = \int_A^B d\lambda \; L(q, \dot{q}) \tag{49}$$

Suppose we have found a path $q(\lambda)$ that minimizes the action, that path would satisfy the following condition:

$$\begin{cases} \text{all small deviation } \delta q \text{ should produce a vanishing first order in the action} : I[q] = I[q + \delta q] + \mathcal{O}(\delta q^2) \\ \text{the correction should vanish at the end points} : \delta q(\lambda_a) = \delta q(\lambda_b) = 0 \end{cases}$$

Under these assumptions:

$$I[q+\delta q] = \int_A^B d\lambda\ L(q+\delta q, \dot{q}+\delta\dot{q}) \tag{50}$$

$$\approx \int_A^B d\lambda\ L(q,\dot{q}) + \delta q \frac{\partial L}{\partial q} + \delta\dot{q}\frac{\partial L}{\partial \dot{q}} + \mathscr{O}(\delta q^2) \tag{51}$$

$$= I[q] + \int_A^B d\lambda\ \delta q \frac{\partial L}{\partial q} + \left(\frac{\partial}{\partial \lambda}\left(\delta q \frac{\partial L}{\partial \dot{q}}\right) - \delta q \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) \tag{52}$$

$$= I[q] + \delta q \frac{\partial L}{\partial \dot{q}}\Big|_{\lambda_a}^{\lambda_b} + \int_A^B d\lambda\ \delta q \left(\frac{\partial L}{\partial q} - \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) \tag{53}$$

$$\therefore \frac{\partial L}{\partial q} - \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{54}$$

■

4.3 Shortest path around a sphere

For the polar line element $ds^2 = R^2 d\theta^2 + R^2 \sin\theta d\phi^2$ let's find the shortest path around a sphere.

First, we can re-coordinatize in such a way that A, and B have the same latitude $\phi_A = \phi_B$ and write ϕ as a function of θ. We would then need to find a path $\phi(\theta)$ to minimize s_{AB}

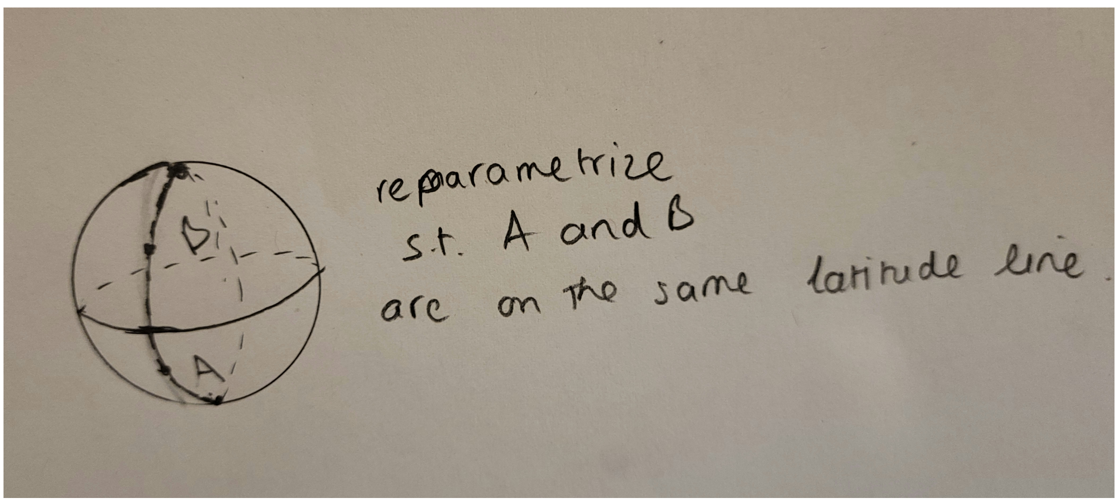


Figure 2: Find the shortest Euclidean distance between A and B on the same latitude line

$$ds = Rd\theta\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2} \tag{55}$$

$$s_{AB} = \int_A^B ds = \int_A^B Rd\theta\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2} \tag{56}$$

$$= \int_A^B Rd\theta\sqrt{1+\phi'^2\sin^2\theta} \tag{57}$$

(58)

Notice that: $\frac{\partial L}{\partial \phi} = 0 \implies \frac{\partial}{\partial \theta}\left(\frac{\partial L}{\partial \phi'}\right) = 0$, hence:

$$\frac{2R\phi'^2 \sin^2\theta}{2\sqrt{1+\phi'^2\sin^2\theta}} = const. \tag{59}$$

$$\frac{\phi'^2 \sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}} = C \tag{60}$$

$$\iff \phi'^2 \sin^2\theta(\sin^2\theta - C^2) = C^2 \tag{61}$$

$$\implies \phi'(\theta) = \pm\frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} \tag{62}$$

SoV and Integrate with respect to θ yields the extremized condition:

$$\cot\theta = \frac{\sqrt{1-C^2}}{C}\cos(\phi - \phi_o) \tag{63}$$

an infant would then see that this is clearly Clairaut's parametrization of a great circle. Therefore, the great circle containing A and B contains both the shortest and the longest path between them

5 Equivalence Principle

The Equivalence Principle is an important idea in relativity regarding each independent observers ability to view them self as "at rest." This relies on the idea that to an observer, acceleration and the presence of a gravitational field can be viewed interchangeably. For example, a person inside a closed box, dropped out of an airplane, could fail to notice the existence of the earths gravitational field because their acceleration from falling counteracts it. In a similar case, a student, studying in Cudahy library would not notice the difference if they were teleported to an identical Cudahy library in space, that was on a rocket, with constant acceleration equal to that of earth's gravitational field.

6 Uniform Acceleration Problem

The problem

Imagine a traveler getting into a spaceship in order to travel to a distant star and then come back to Earth. We assume that the position of the Earth is $x_0 = 0$ and that the time the stationary observer experiences until the astronaut reaches the midpoint of the distance to the star is t_*. (Let's assume that there is a well defined midpoint with some sort of physical landmark, so we avoid the whole length contraction shenanigans, as the two observers will agree on the landmark). The astronaut is experiencing a constant proper acceleration, g, and the observer measures the astronaut's velocity,

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{gt}{\sqrt{1+(\frac{gt}{c})^2}} \tag{64}$$

We divide the trip into four parts, from Earth to the midpoint to the star, the rest of the travel to the star, the return trip up to the midpoint and from there back to Earth. In order for the astronaut to return to Earth, we will assume that turning the rocket around happens instantaneously at the points where the acceleration should change direction.
Find the acceleration function in the four parts and using that, find the velocity and position functions from the point of view of the stationary observer on Earth.

The solution

When we take the derivative of (1) with respect to dt, we have:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{gt}{\sqrt{1+(\frac{gt}{c})^2}} \Rightarrow$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{g}{\sqrt{1+(\frac{gt}{c})^2}} + \frac{\frac{-g^3t^2}{c^2}}{\left(\sqrt{1+(\frac{gt}{c})^2}\right)^3} \Rightarrow$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{g}{\sqrt{1+(\frac{gt}{c})^2}}\left(1 - \frac{(\frac{gt}{c})^2}{1+(\frac{gt}{c})^2}\right) \Rightarrow$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{g}{\left(1+(\frac{gt}{c})^2\right)^{3/2}} \tag{65}$$

This is the equation for acceleration for the first quarter of the trip. When we look at how this is affected when the astronaut needs to decelerate at $t = t_*$, we see that we have to make two changes. First, flip the sign as the acceleration is now in the opposite direction. Then, because of the symmetry (look at the graph of acceleration in the end) of the acceleration function, we need to do a translation in time, $t \to t - 2t_*$. Thus, for the first two quarters of the trip, the acceleration function looks like:

$$a_1(t) = \frac{g}{\left(1+(\frac{gt}{c})^2\right)^{3/2}}, 0 \leq t \leq t_* \tag{66}$$

$$a_2(t) = \frac{-g}{\left(1+(\frac{g(t-2t_*)}{c})^2\right)^{3/2}}, t_* \leq t \leq 2t_* \tag{67}$$

By integrating the above function over dt, we get that the velocity for the first two quarters is:

$$v_1(t) = \frac{gt}{\sqrt{1+(\frac{gt}{c})^2}}, 0 \leq t \leq t_* \tag{68}$$

$$v_2(t) = \frac{-g(t-2t_*)}{\sqrt{1+(\frac{gt}{c})^2}} + D, t_* \leq t \leq 2t_* \tag{69}$$

In order to find the integration constant we have to consider the fact that $v_1(t_*) = v_2(t_*)$, since the velocity should be a continuous function.

By plugging in $t = t_*$ it is fairly trivial to see that $D = 0$.

In order to get the first two quarters' position functions we need to integrate once more over dt, in which case we get:

$$x_1(t) = \frac{c^2}{g}\sqrt{1+\left(\frac{gt}{c}\right)^2} - \frac{c^2}{g}, 0 \leq t \leq t_* \tag{70}$$

$$x_2(t) = -\frac{c^2}{g}\sqrt{1+\left(\frac{g(t-2t_*)}{c}\right)^2} + E, t_* \leq t \leq 2t_* \tag{71}$$

In order to find the integration constant, E, we again need to consider that $x_1(t_*) = x_2(t_*)$.

This gives us that : $E = \frac{c^2}{g}\left(2\sqrt{1+\left(\frac{gt_*}{c}\right)^2} - 1\right)$

By plugging that in (63), we can see that:

$$x_2(t) = -\frac{c^2}{g}\sqrt{1+\left(\frac{g(t-2t_*)}{c}\right)^2} + \frac{c^2}{g}\left(2\sqrt{1+\left(\frac{gt_*}{c}\right)^2}-1\right) \tag{72}$$

In order to find the acceleration, velocity and position functions for the last two quarters of the trip, we apply a similar analysis.

$$a_3(t) = \frac{-g}{\left(1+\left(\frac{g(t-2t_*)}{c}\right)^2\right)^{3/2}}, 2t_* \leq t \leq 3t_* \tag{73}$$

$$a_4(t) = \frac{g}{\left(1+\left(\frac{g(t-4t_*)}{c}\right)^2\right)^{3/2}}, 3t_* \leq t \leq 4t_* \tag{74}$$

It is expected that $a_2(t) = a_3(t)$, since for that time period the deceleration turn into an acceleration towards the opposite direction as soon as $v = 0$ at $t = 2t_*$.

$$v_3(t) = \frac{-g(t-2t_*)}{\sqrt{1+(\frac{gt}{c})^2}}, 2t_* \leq t \leq 3t_* \tag{75}$$

$$v_4(t) = \frac{g(t-4t_*)}{\sqrt{1+(\frac{g(t-4t_*)}{c})^2}}, 3t_* \leq t \leq 4t_* \tag{76}$$

Lastly, one more integration will give us the position functions:

$$x_3(t) = -\frac{c^2}{g}\sqrt{1+\left(\frac{g(t-2t_*)}{c}\right)^2} - \frac{c^2}{g}\left(2\sqrt{1+\left(\frac{gt}{c}\right)^2}-1\right) \tag{77}$$

$$x_4(t) = -\frac{c^2}{g}\sqrt{1+\left(\frac{g(t-4t_*)}{c}\right)^2} + K \tag{78}$$

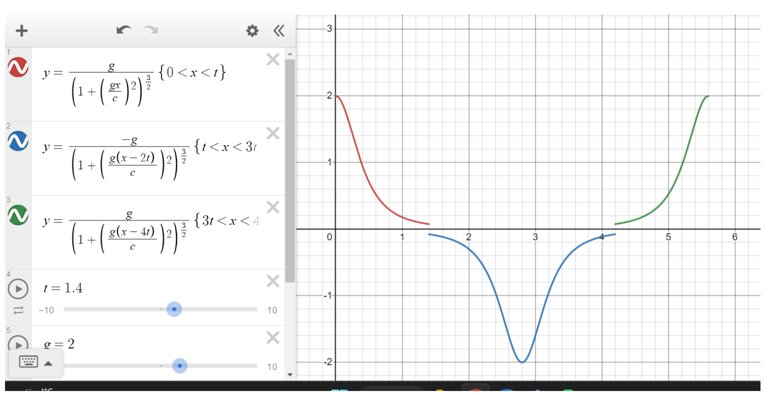
We can find K by requiring that $x_3(3t_*) = x_4(t_*)$.

Then, a straightforward calculation gives us $K = -\frac{c^2}{g}\left(-2\sqrt{1+\left(\frac{gt_*}{c}\right)^2}+1\right)$

Thus,

$$x_4(t) = -\frac{c^2}{g}\sqrt{1+\left(\frac{g(t-4t_*)}{c}\right)^2} - \frac{c^2}{g}\left(-2\sqrt{1+\left(\frac{gt_*}{c}\right)^2}+1\right) \tag{79}$$

The graph below shows the acceleration function plotted in time, where the parameter t is equivalent to our t_*.



7 Another Uniform Acceleration Problem

Consider the transformations:

$$t = \Big(\frac{c}{g} + \frac{x'}{c}\Big)\sinh\Big(\frac{gt'}{c}\Big)$$
$$x = c\Big(\frac{c}{g} + \frac{x'}{c}\Big)\cosh\Big(\frac{gt'}{c}\Big) - \frac{c^2}{g}$$
$$y = y', z = z'$$

Then, the line element, ds^2 is given by:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{80}$$

$$dt = \frac{\partial t}{\partial x'}dx' + \frac{\partial t}{\partial t'}dt' = \frac{1}{c}\sinh\Big(\frac{gt'}{c}\Big)dx' + \frac{g}{c}(\frac{c}{g} + \frac{x'}{c})\cosh\Big(\frac{gt'}{c}\Big)dt' \Rightarrow$$
$$dt = \frac{1}{c}\sinh\Big(\frac{gt'}{c}\Big)dx' + (1 + \frac{gx'}{c^2})\cosh\Big(\frac{gt'}{c}\Big)dt' \tag{81}$$

Similarly,

$$dx = \cosh\Big(\frac{gt'}{c}\Big)dx' + c(1 + \frac{gx'}{c^2})\sinh\Big(\frac{gt'}{c}\Big)dt' \tag{82}$$

Then, the line element then (ignoring the dy, dz terms) reduces to:

$$\begin{aligned}
ds^2 = -c^2\Bigg[&\frac{1}{c^2}\sinh^2\Big(\frac{gt'}{c}\Big)dx'^2 + \Big(1 + \frac{gx'}{c^2}\Big)^2\cosh^2\Big(\frac{gt'}{c}\Big)dt'^2 + \frac{2}{c}\Big(1 + \frac{gx'}{c^2}\Big)\sinh\Big(\frac{gt'}{c}\Big)\cosh\Big(\frac{gt'}{c}\Big)dx'dt'\Bigg] \\
&+ \cosh^2\Big(\frac{gt'}{c}\Big)dx'^2 + c^2\Big(1 + \frac{gx'}{c^2}\Big)^2\sinh^2\Big(\frac{gt'}{c}\Big)dt'^2 - 2c\Big(1 + \frac{gx'}{c^2}\Big)\sinh\Big(\frac{gt'}{c}\Big)\cosh\Big(\frac{gt'}{c}\Big)dx'dt' \\
&= -\sinh^2\Big(\frac{gt'}{c}\Big)dx'^2 - c^2\Big(1 + \frac{gx'}{c^2}\Big)\cosh^2\Big(\frac{gt'}{c}\Big)dt'^2 + \cosh^2\Big(\frac{gt'}{c}\Big)dx'^2 + c^2\Big(1 + \frac{gx'}{c^2}\Big)^2\sinh^2\Big(\frac{gt'}{c}\Big)dt'^2 \\
&= -c^2\Big(1 + \frac{gx'}{c^2}\Big)^2 dt'^2 + dx'^2
\end{aligned}$$

If we Taylor expand the transformations up to second order, we can see that they resemble Newtonian Mechanics for acceleration equal to g.

$$\cosh\Big(\frac{gt'}{c}\Big) \approx 1 + \frac{g^2t'^2}{2c^2}, \sinh\Big(\frac{gt'}{c}\Big) \approx \frac{gt'}{c} \tag{83}$$

And thus: $t \approx t'$ and $x \approx x' + \frac{1}{2}gt'^2$

Lastly, since the line element term of dx' has no time dependence, we can conclude that, in time, the height any object moving in a way that obeys the above transformation will remain the same, and thus the body will be rigid.

8 Lorentz Transformations

8.1 4-Vectors

$$\vec{r} \to r^i \qquad i = 1, 2, 3 \qquad \vec{r} = (r^1, r^2, r^3) \tag{84}$$

$$\vec{A}\cdot\vec{B} \to A^iB^i = \sum_i A^iB^i = A^1B^1 + A^2B^2 + A^3B^3 \tag{85}$$

$$= A_iB^i = (A^1 \quad A^2 \quad A^3)\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{86}$$

$$M_i^j = \begin{pmatrix} M_1^1 & M_2^1 & M_3^1 \\ M_1^2 & M_2^2 & M_3^2 \\ M_1^3 & M_2^3 & M_3^3 \end{pmatrix} \tag{87}$$

Matrix mechanics follow as usual with this shorthand notation of upper and lower indices.
Using this 4-vector notation, we intoduce a position vector $X^{\mu} \quad \mu = 0, 1, 2, 3$ Where

$$X^{\mu} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \vec{r} \end{pmatrix} \text{ and } \Delta X^{\mu} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} c\Delta t \\ \vec{r} \end{pmatrix} \tag{88}$$

To simplify calculations involving 4-vectors we introduce the Minkowski metric

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{89}$$

With this useful tool,

$$(\Delta x)_{\mu}(\Delta)^{\mu} = (-c\Delta t, \Delta\vec{r}) \begin{pmatrix} c\Delta t \\ \Delta\vec{r} \end{pmatrix} = \eta_{\mu\nu}(\Delta x)^{\mu}(\Delta x)^{\nu} \tag{90}$$

8.2 Boosts, Rotations and and translations

In Relativity, there are 3 types of continuous symmetries that leave the line element, Δs^2, invariant. First, we have the Lorentz boosts, which are described by the following equation:

$$x^{'\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{91}$$

where, for a boost in the x- direction:

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & \frac{-\gamma v}{c^2} & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{92}$$

Second, we have spatial and time translations, which are simply described: $\vec{r} \rightarrow \vec{r} + \vec{a}$ and $t \rightarrow t + b$
Lastly, we have the spatial rotations around x,y and z, which are described by the known rotation matrices.

9 Relativistic Formulation of Maxwell Equations

9.1 Sources of Electric and Magnetic Fields

Recall that Maxwell's Equations relate the electric and magnetic fields to their sources (charge and current density, respectively).

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{93}$$

$$\nabla \cdot \vec{B} = 0 \tag{94}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{95}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{96}$$

These sources will change with reference frame, and as a result, we must construct a method that represents the electric and magnetic fields while considering relativistic phenomena. We can begin by representing the sources for these fields as a 4- vector:

$$J^{\mu} = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \tag{97}$$

0-th index is the charge density, and the remaining three are the three components of the current density. The factor of c is added to maintain units. This four-vector is covariant, in that it transforms by the Minkowski rank 0,2 tensor $\eta_{\mu\nu}$ as follows (as seen in Eqn. 90):

$$J_\mu J^\mu = \eta_{\mu\nu} J^\nu J^\mu \tag{98}$$

where $\eta_{\mu\nu}$ is given in Eqn. 89.

The continuity equation $\frac{\partial \rho}{\partial \tau} + \mathbf{\nabla} \cdot \vec{J} = 0$ can thus be rewritten using relativistic notation. Instead of a derivative, we use a 4-vector differential operator that also behaves covariantly, ∂_μ. It acts so that:

$$\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t} \tag{99}$$

$$\partial_i = \frac{\partial}{\partial x^i} \tag{100}$$

Thus

$$\begin{aligned} \partial_\mu J^\mu &= \partial_0 J^0 + \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 && (101) \\ &= \frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \frac{\partial}{\partial x}(J_x) + \frac{\partial}{\partial y}(J_y) + \frac{\partial}{\partial z}(J_z) && (102) \\ &= \frac{\partial \rho}{\partial t} + \mathbf{\nabla} \cdot \vec{J} && (103) \end{aligned}$$

And so, the relativistic formulation of the continuity equation is $\partial_\mu J^\mu = 0$. Conservation of charge.

9.2 Potential 4-Vectors

Now that we have a relativistic formulation of the sources for electric and magnetic fields, the next step is to proceed to a formulation that relates these sources to the fields. However, $\vec{E}$ and $\vec{B}$ can't be represented as 4-vectors themselves because they are three-dimensional vectors in classical physics. Instead, we will create a 4-vector combining the electric potential Φ, a scalar, and the magnetic potential $\vec{A}$, a quantity that is a vector in classical electrodynamics.

$$A^\mu = \begin{pmatrix} \frac{\Phi}{c} \\ \vec{A} \end{pmatrix} \tag{104}$$

Then $\vec{E}$ and $\vec{B}$ become

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{105}$$

$$\vec{B} = -\mathbf{\nabla} \times \vec{A} \tag{106}$$

As Φ and $\vec{A}$ are potentials, they are non-unique. In other words, the potentials can be shifted by a *gauge factor* λ, and they will produce the same electric and magnetic fields.

$$\vec{A} \rightarrow \vec{A} + \mathbf{\nabla} \cdot \lambda \tag{107}$$

$$\Phi \rightarrow \Phi + \partial_0 \lambda \tag{108}$$

We now have the framework necessary for a relativistic representation of $\vec{E}$ and $\vec{B}$ using tensors.

9.3 Maxwell Field Strength Tensor

As mentioned earlier, $\vec{E}$ and $\vec{B}$ are both vectors and thus have too much information to represent in a 4-vector. Instead, we will use the Maxwell Field Strength Tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{109}$$

Note that, as it represents $\vec{E}$ and $\vec{B}$, $F_{\mu\nu}$ doesn't change with the choice of gauge.

$$\partial_\mu \partial_\nu \lambda = \partial_\nu \partial_\mu \lambda \tag{110}$$

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda) \\ &= \partial_\mu A_\nu + \partial_\mu \partial_\nu \lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \lambda \\ &= F_{\mu\nu} \end{aligned}$$

Let us examine the tensor in more detail. First, this is an anti-symmetric rank 0,2 tensor. Anti-symmetric means that $F_{\mu\nu} = -F_{\nu\mu}$. Furthermore, when represented in matrix form, we can see that $F_{\mu\nu}$ contains six unique pieces of information - three for $\vec{E}$ and three for $\vec{B}$.

$$\begin{pmatrix} 0 & a & b & c \\ & 0 & d & e \\ & & 0 & f \\ & & & 0 \end{pmatrix} \tag{111}$$

This is evident when we calculate individual components:

$$F_{01} = \frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial}{\partial x}\left(-\frac{\Phi}{c}\right) = \frac{1}{c}\left(\frac{\partial \Phi}{\partial t} + \frac{\partial A_x}{\partial t}\right) = -\frac{E_x}{c} \tag{112}$$

$$F_{12} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\boldsymbol{\nabla} \times \vec{A})_z = B_z \tag{113}$$

Where ε_{ijk} is the The components of $\vec{E}$ and $\vec{B}$ are thus represented as follows:

$$F_{0i} = -\frac{E_i}{c} \tag{114}$$

$$F_{ij} = \varepsilon_{ijk} B_k \tag{115}$$

We can move the indices up, resulting in:

$$F^{0i} = \frac{E^i}{c} \tag{116}$$

$$F^{ij} = \varepsilon^{ijk} B^k \tag{117}$$

9.4 Maxwell's Equations

Two of Maxwell's Equations imply the existence of potentials for the fields:

$$\boldsymbol{\nabla} \times \vec{E} = 0 \Rightarrow \nabla \Phi = \vec{E} \tag{118}$$

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \Rightarrow \boldsymbol{\nabla} \times \vec{A} = \vec{B} \tag{119}$$

So, we have two remaining Maxwell Equations to recover: $\boldsymbol{\nabla} \cdot \vec{E}$ and $\boldsymbol{\nabla} \times \vec{B}$. To do this, we will look at the derivatives of $\vec{E}$ and $\vec{B}$, which should be related to the sources, which live in the 4-vector A^μ, by applying the 4-vector differential operator ∂_μ to the Maxwell Field Strength Tensor $F^{\mu\nu}$. We can use split the differential operator into two parts"

$$\partial_\mu F^{\mu\nu} = \partial_0 F^{0\nu} + \partial_i F^{i\nu} \tag{120}$$

We can apply it to $\nu = 0$:

$$\partial_\mu F^{\mu 0} = \partial_0 F^{00} + \partial_i F^{i0} \tag{121}$$

$$= -\frac{1}{c}\partial_i E^i \tag{122}$$

$$= -\frac{1}{c}\left(\boldsymbol{\nabla} \cdot \vec{E}\right) \tag{123}$$

$$= -\frac{1}{c}\frac{\rho}{\epsilon_0} \tag{124}$$

$$= -\mu_0 J^0 \tag{125}$$

This is the relativistic formulation of Coulomb's law. To get the other formulation, we look at $\nu = i$ for $i = 1, 2, 3$ as is convention.

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} \tag{126}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}E^i\right) + \partial_j(\varepsilon^{ijk}B^k) \tag{127}$$

$$= \frac{1}{c^2}\frac{\partial E^i}{\partial t} - \varepsilon^{ijk}\partial_j B^k \tag{128}$$

$$= \frac{1}{c^2}\frac{\partial E^i}{\partial t} - (\boldsymbol{\nabla} \times \vec{B})_i \tag{129}$$

$$= -\mu_i J^i \tag{130}$$

Thus the Maxwell's Equations can be rewritten using relativistic notation as:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{131}$$

$$\tag{132}$$